Supplementary material for "Estimation and reduced bias estimation of the residual dependence index with unnamed marginals"

Jennifer Israelsson

Department of Mathematics and Statistics, University of Reading, UK,

Emily Black

Department of Meteorology, University of Reading, UK,

Cláudia Neves

Department of Mathematics, King's College London, UK

and

David Walshaw

School of Mathematics & Statistics, Newcastle University, UK

A Simulation results

In the interest of completeness, this section is devoted to presenting simulation results on the basis of four copula families outlined below.

(i) **Farlie-Gumbel-Morgenstern**: $\mathcal{C}_\theta(u,v) = uv\{1+\theta(1-u)(1-v)\}$, $(u,v) \in [0,1]^2$, $\theta \in [-1,1]$. Hence, with $\theta > -1$,

$$\frac{P(1-F_1(X) < tx, 1-F_2(Y) < ty)}{P(1-F_1(X) < t, 1-F_2(Y) < t)} = xy\Big\{1 - \frac{\theta t}{1+\theta}(x+y-2) + O(t^2)\Big\},$$

as $t \to \infty$, yields $\eta = \tau = 1/2$ in (14).

(ii) **Frank distribution** with copula function

$$\mathcal{C}_\theta(u,v) = -\frac{1}{\theta}\log\Big(1 - \frac{(1-e^{-\theta u})(1-e^{-\theta v})}{1-e^{-\theta}}\Big), \quad (u,v) \in [0,1]^2,\ \theta > 0,$$

satisfying the second order condition (14) with $\eta = \tau = 1/2$.

(iii) **Ali-Mikhail-Had distribution**, whose copula function is given by

$$\mathcal{C}_\theta(u,v) = \frac{u\,v}{1-\theta(1-u)(1-v)},\ (u,v)\in[0,1]^2,\ \theta\in[-1,1].$$

For $\theta=-1$ the second order condition (14) is satisfied with $\eta = 1/3$ and $\tau = 2\eta = 2/3$.

(iv) **Bivariate Normal**: $\mathcal{C}_\theta(u,v) = \Phi_\theta\big(\phi^{-1}(u),\phi^{-1}(v)\big)$ with Φ_θ the bivariate standard normal distribution function with correlation $\theta\in[-1,1]$ and Φ the scalar standard normal distribution function. Although the bivariate normal distribution with $\theta<1$ satisfies (13) with $\eta=\frac{1+\theta}{2}$, the resutling $\tau=0$ deems it out of scope in to relation the setting laid by (14) for this paper. Nonetheless, we choose to include it for assessing robustness.

Figures 7 to 10 display finite sample behaviour of estimators $\hat{\eta}_q$, introduced in (10), and the new class of estimators $\hat{\eta}_q^{(S)}$ given by (12) with $(a,b)=(1/p,1/q-1)$ in the following way. Performance assessment of the estimators is based on the estimated mean (on the left hand-side) of each Figure 7 to 10, and estimated mean squared error (MSE) (on the right hand-side), as a function of the top sample fraction $k/n \leq 0.3$. The parameter q takes values 0.1(0.1)1.9. The Hill estimator is retrieved for $q=1$. The aim of this section is to bear out the asymptotic equivalence established in Theorem 3 through finite sample behaviour of the relevant estimators by drawing on the four copula models above. All Figures conform to that transformation to Pareto or Fréchet with the shift by 1/2 determines virtually the same findings for this class of estimators for the residual dependence index $\eta\in[0,1)$. There are instances at which the shifted Fréchet transformation leads to marginal improvement, for example in case of the underlying Farlie-Gumbel-Morgenstern copula, but this is not always the case. Taking everything into account, it seems reasonable to conclude that this choice of marginals does not play a role eventually, for the general class of estimators (8). Moreover, a selection of $q<1$ should take preference if there is preliminary evidence that the true value η is greater than 1/2, whereas a value $q>1$ should be specified in connection with potential $\eta<1/2$. The intermediate case of near independence $\eta=1/2$ is quite mixed, but it is also in this case that the reduced bias estimators seem to have best performance, thus making a choice of q less relevant.

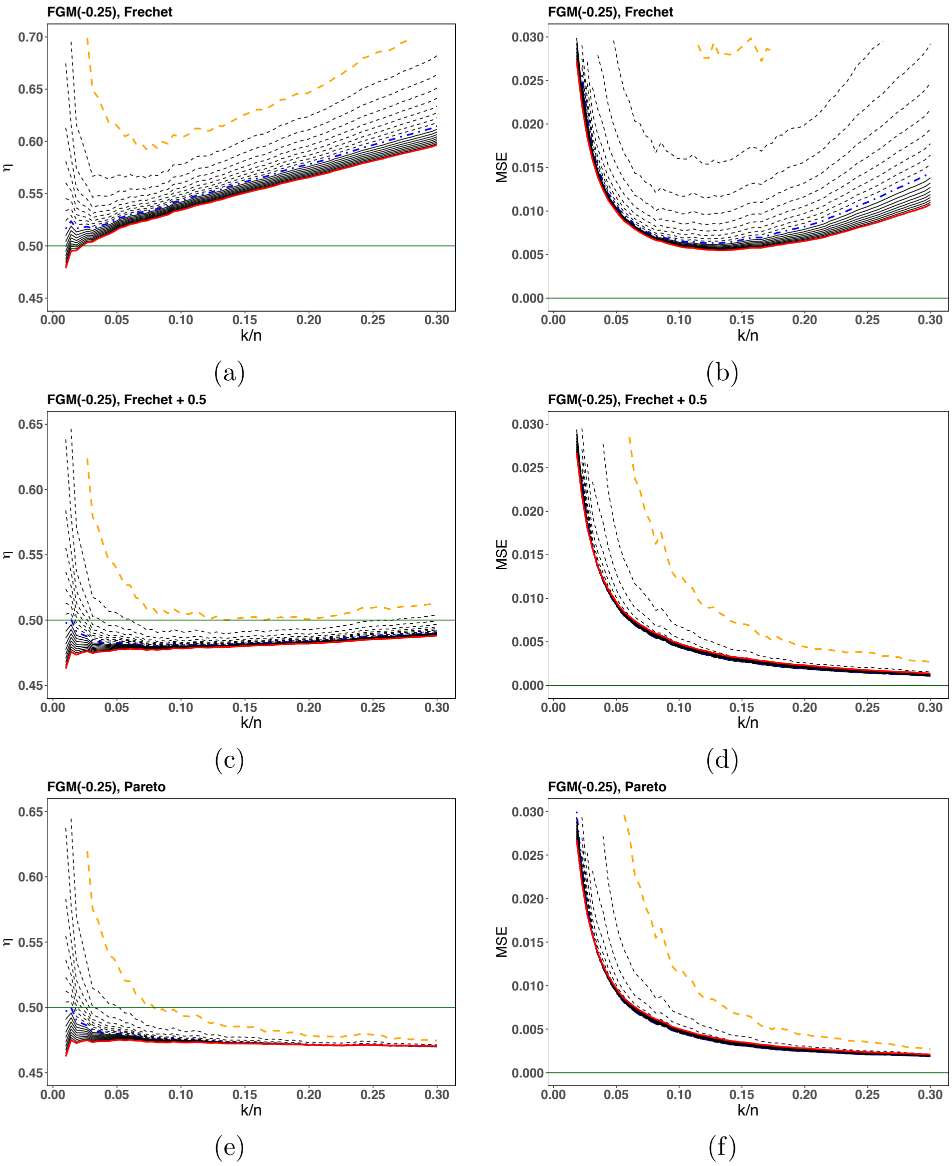

Figure 7: **Farlie-Gumbel-Morgenstern copula with** $\theta = -0.25$ ($\eta = \tau = 1/2$). Plots (a) and (b) stem from unit Fréchet marginals; (c) and (d) result from adding the location shift; plots (e) and (f) correspond to the typical standard Pareto marginals. Dashed lines identify $q < 1$ with orange for $q = 0.1$, solid lines represent $q > 1$ with red for $q = 1.9$. The blue dash-dotted line in between ($q = 1$) corresponds to the Hill estimator.

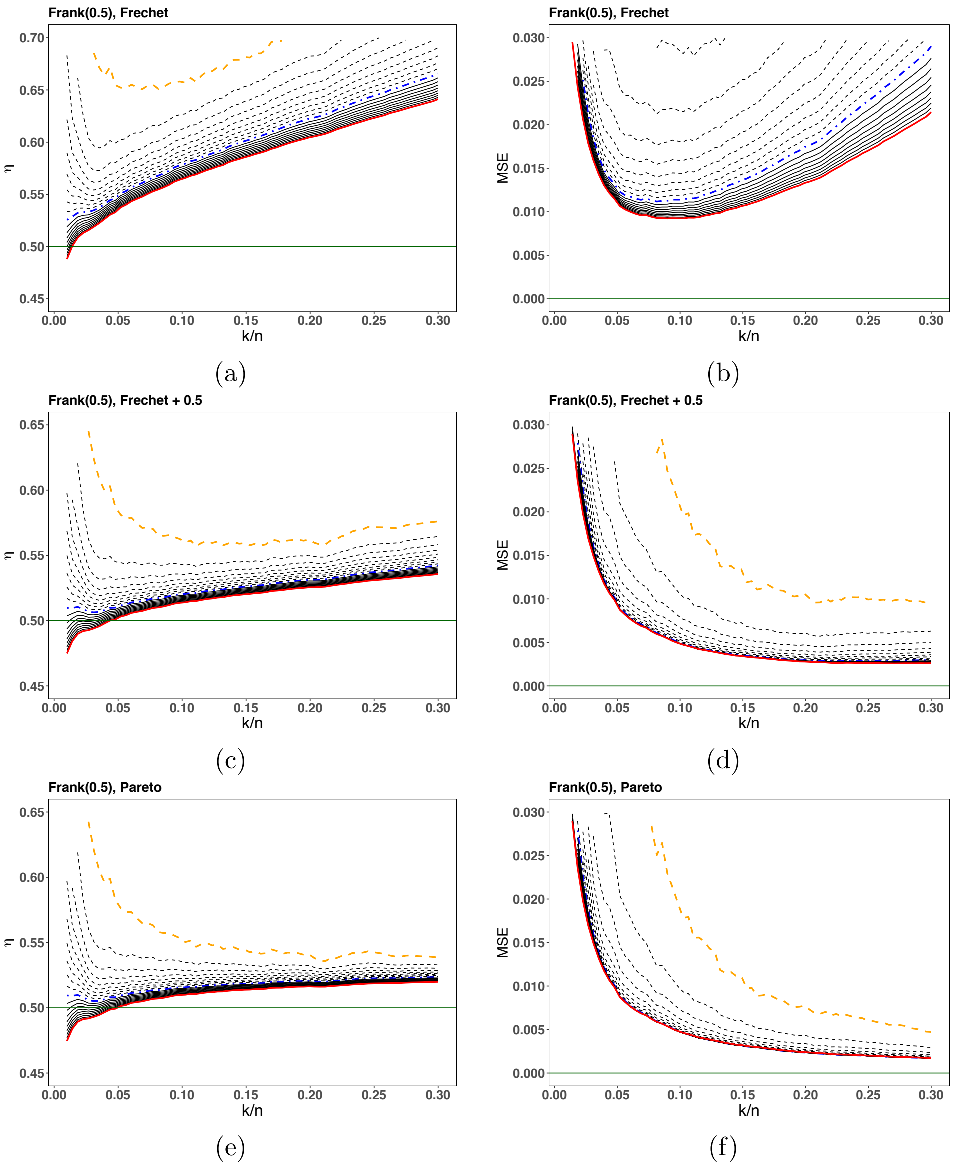

Figure 8: **Frank copula with** $\theta = 0.5$ ($\eta = \tau = 1/2$). Plots (a) and (b) stem from unit Fréchet marginals; (c) and (d) result from adding the location shift; plots (e) and (f) correspond to the typical standard Pareto marginals. Dashed lines identify $q < 1$ with orange for $q = 0.1$, solid lines represent $q > 1$ with red for $q = 1.9$. The blue dash-dotted line in between ($q = 1$) corresponds to the Hill estimator.

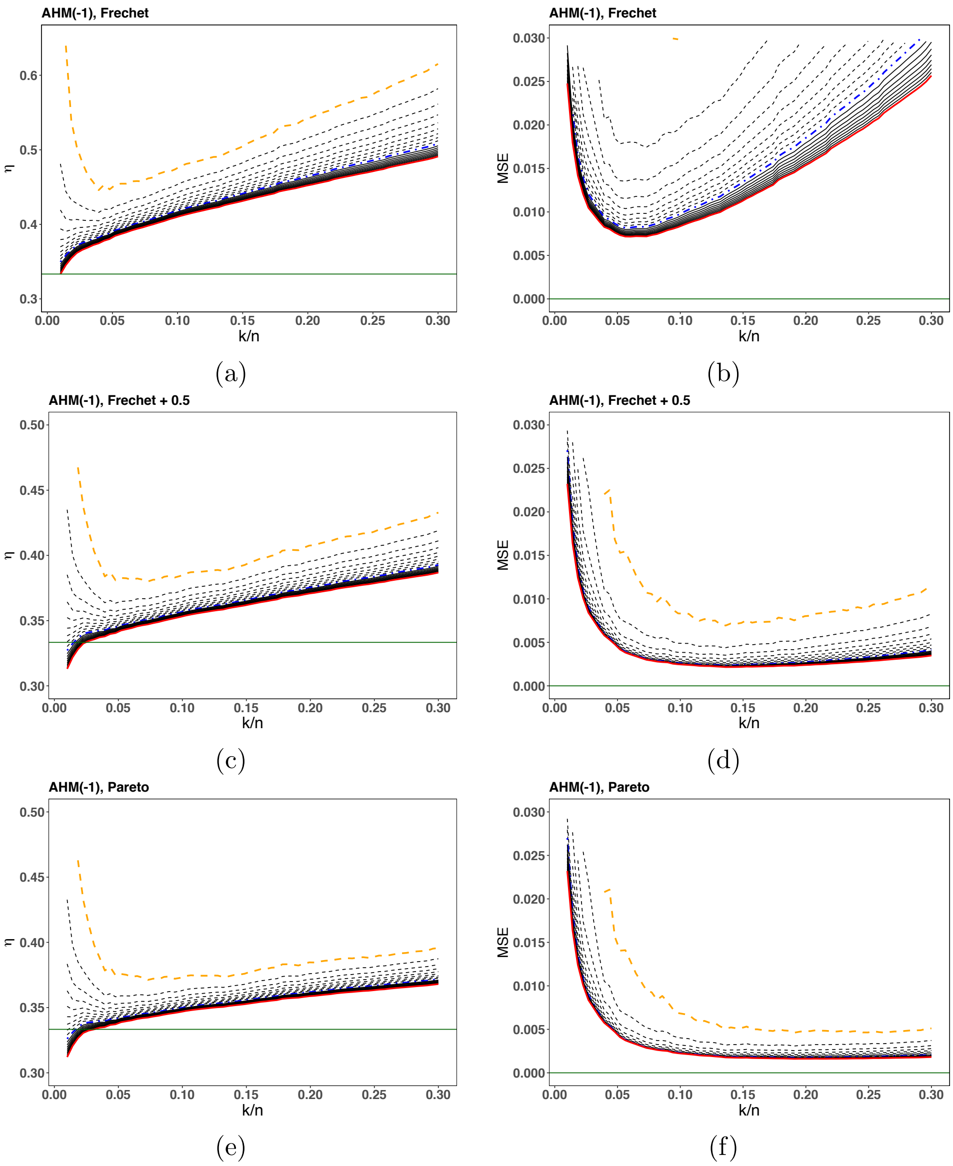

Figure 9: **Ali-Mikhail-Haq copula with** $\theta = -1$ ($\eta = 1/3$, $\tau = 2/3$). Plots (a) and (b) stem from unit Fréchet marginals; (c) and (d) result from adding the location shift; plots (e) and (f) correspond to the typical standard Pareto marginals. Dashed lines identify $q < 1$ with orange for $q = 0.1$, solid lines represent $q > 1$ with red for $q = 1.9$. The blue dash-dotted line in between ($q = 1$) corresponds to the Hill estimator.

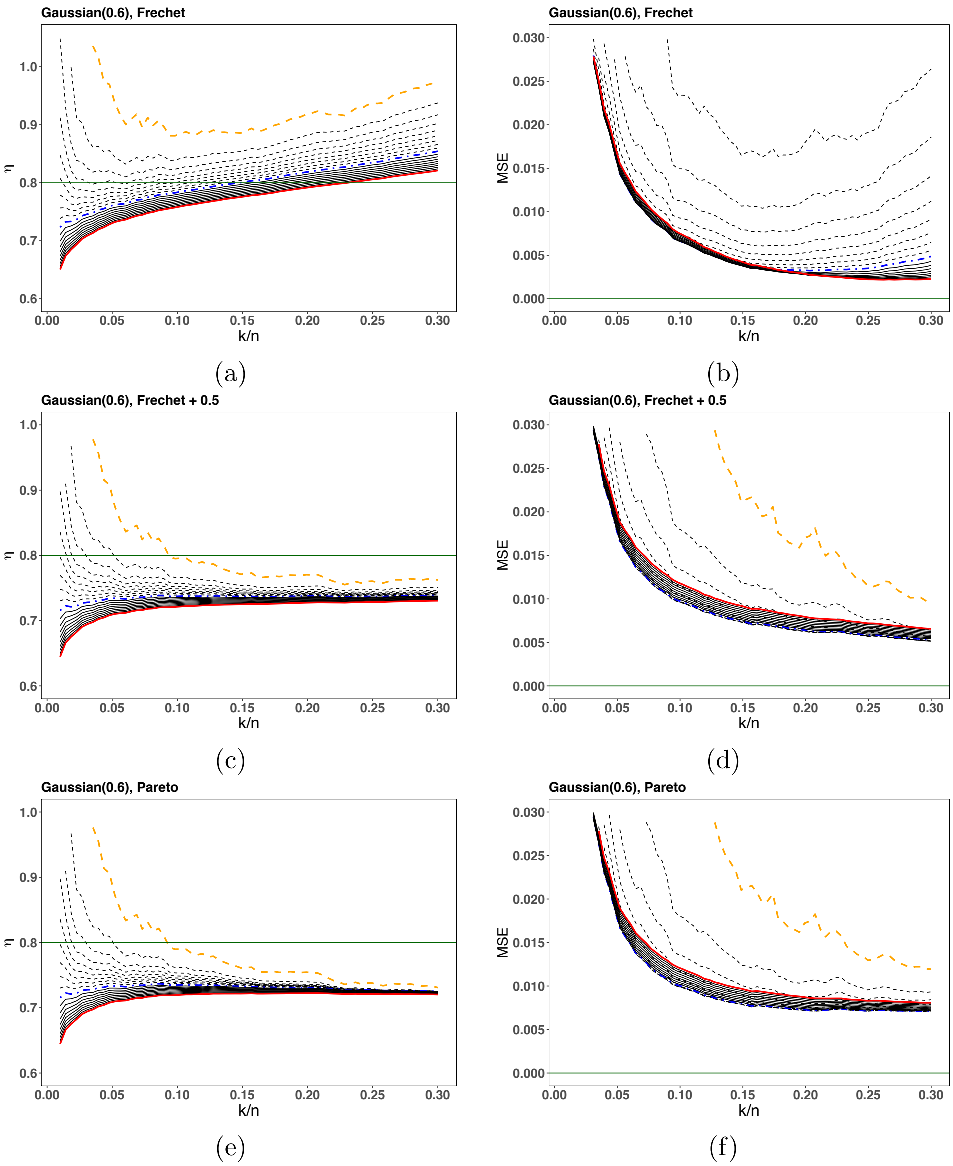

Figure 10: **Gaussian copula with** $\theta = 0.6$ $(\eta = 0.8, \tau = 0)$. Plots (a) and (b) stem from unit Fréchet marginals; (c) and (d) result from adding the location shift; plots (e) and (f) correspond to the typical standard Pareto marginals. Dashed lines identify $q < 1$ with orange for $q = 0.1$, solid lines represent $q > 1$ with red for $q = 1.9$. The blue dash-dotted line in between $(q = 1)$ corresponds to the Hill estimator.

B Further simulation results

The following results correspond to the mean-of-order p class of estimators, corresponding to the parametrisation $(a,b) = (1/(1-p),\, q-1)$ with $1/p + 1/q = 1$ in both (10) and (12). This section aims to demonstrate that the simulation results are more discrepant and less consistent than those stemming from the leading parametrisation $(a,b) = (p^{-1},\, q^{-1} - 1)$, in the sense that values of q well above 1 or much lower than 1 are often required in order to minimise bias, and thereby attaining optimal mean squared error.

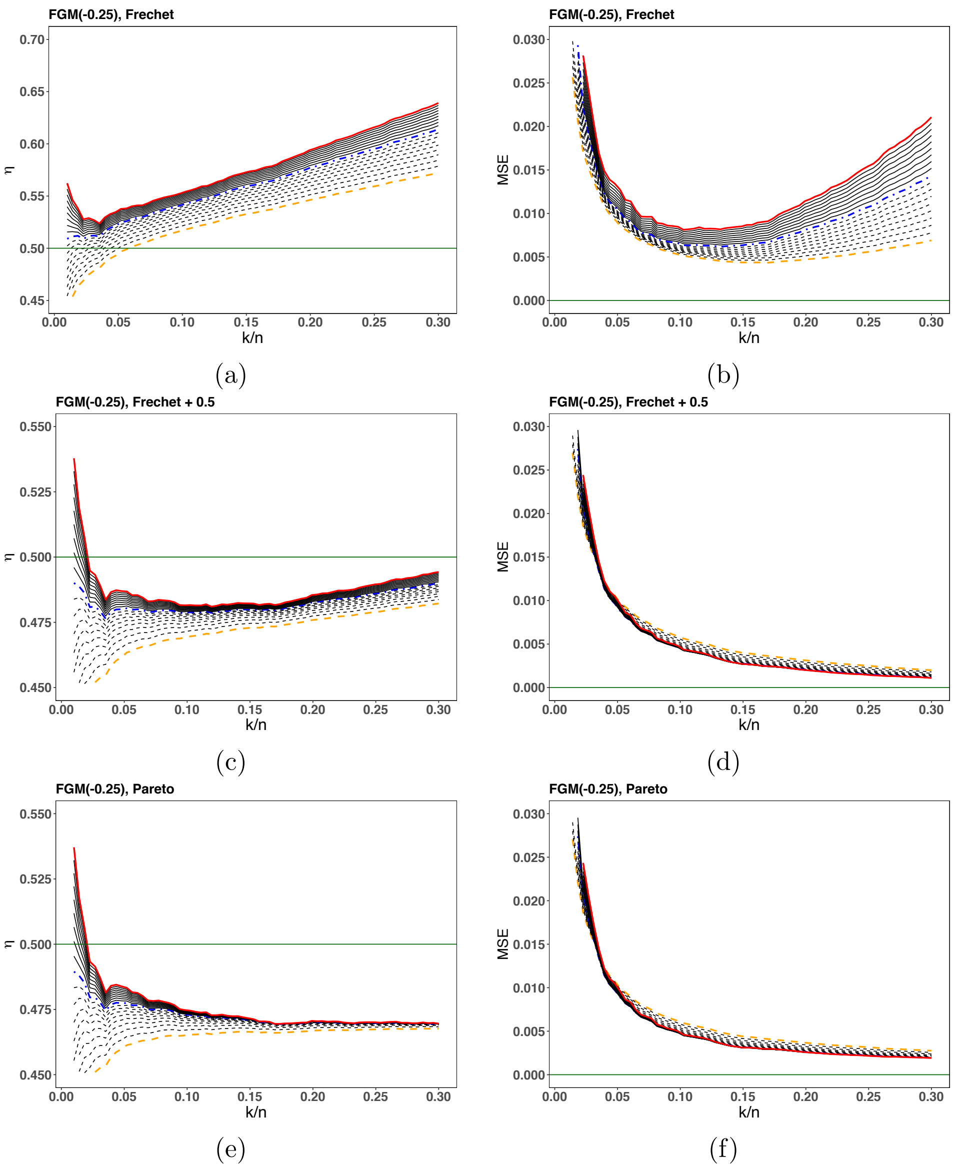

Figure 11: **Farlie-Gumbel-Morgenstern copula with** $\theta = -0.25$ ($\eta = \tau = 1/2$). Plots (a) and (b) stem from unit Fréchet marginals; (c) and (d) result from adding the location shift; plots (e) and (f) correspond to the typical standard Pareto marginals. Dashed lines identify $q < 1$ with orange for $q = 0.1$, solid lines represent $q > 1$ with red for $q = 1.9$. The blue dash-dotted line in between ($q = 1$) corresponds to the Hill estimator.

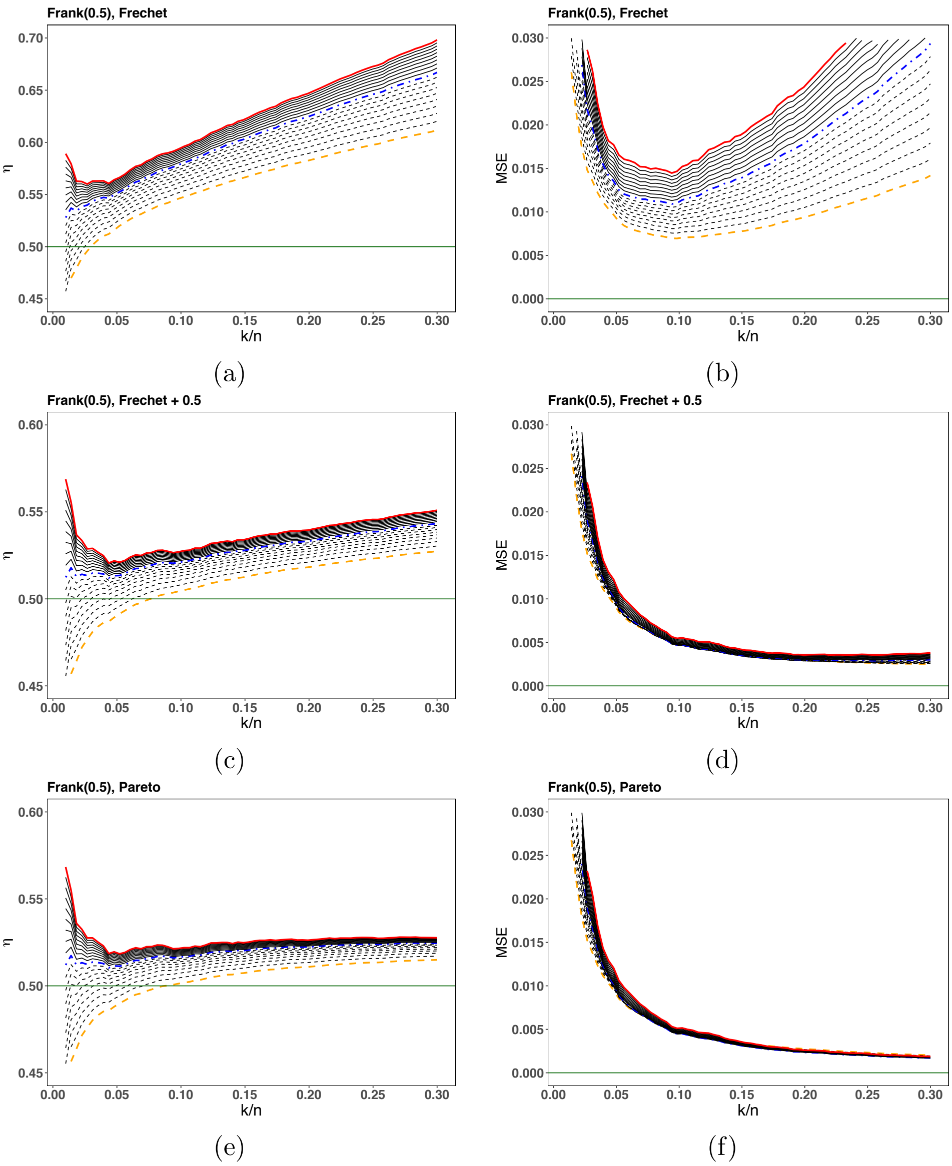

Figure 12: **Frank copula with** $\theta = 0.5$ $(\eta = \tau = 1/2)$. Plots (a) and (b) stem from unit Fréchet marginals; (c) and (d) result from adding the location shift; plots (e) and (f) correspond to the typical standard Pareto marginals. Dashed lines identify $q < 1$ with orange for $q = 0.1$, solid lines represent $q > 1$ with red for $q = 1.9$. The blue dash-dotted line in between $(q = 1)$ corresponds to the Hill estimator.

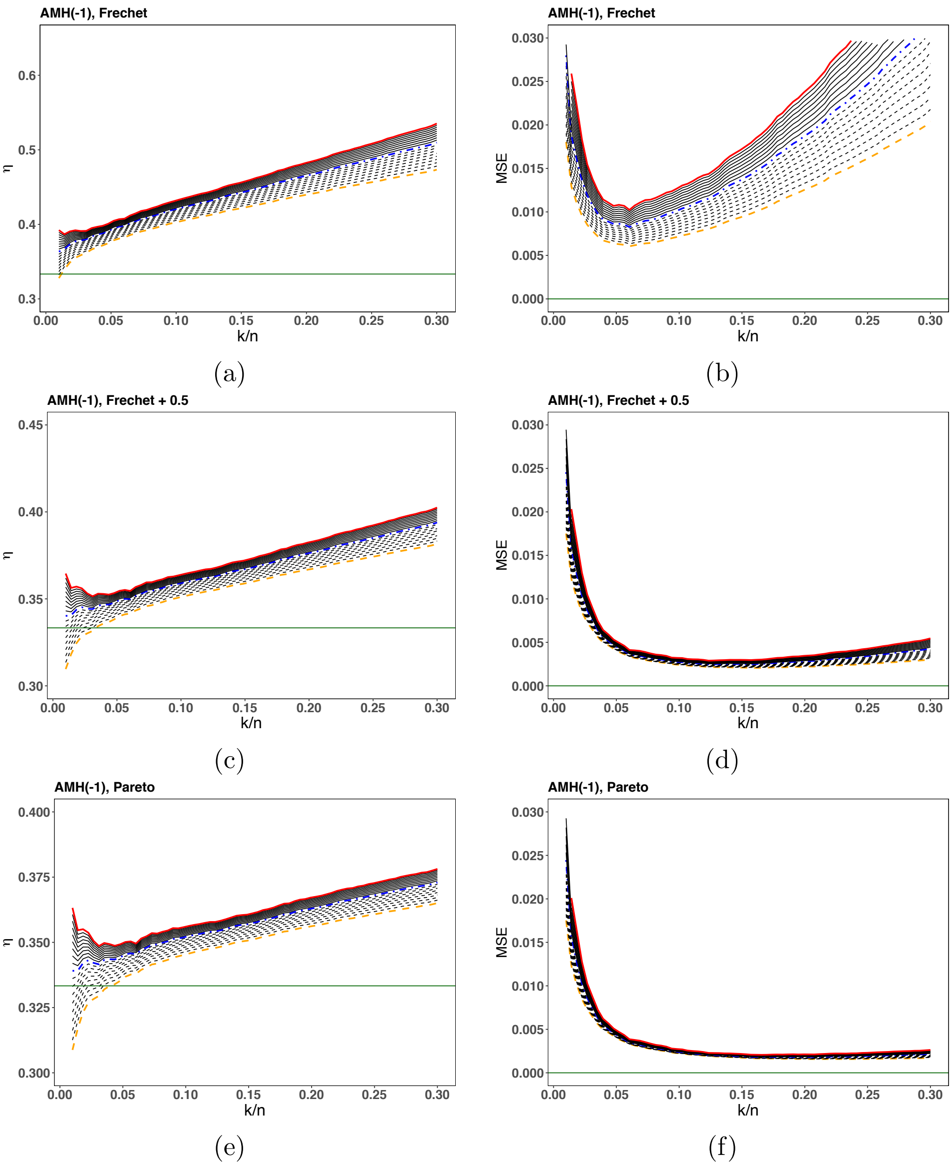

Figure 13: **Ali-Mikhail-Haq copula with** $\theta = -1$ ($\eta = 1/3$, $\tau = 2/3$). Plots (a) and (b) stem from unit Fréchet marginals; (c) and (d) result from adding the location shift; plots (e) and (f) correspond to the typical standard Pareto marginals. Dashed lines identify $q < 1$ with orange for $q = 0.1$, solid lines represent $q > 1$ with red for $q = 1.9$. The blue dash-dotted line in between ($q = 1$) corresponds to the Hill estimator.